

Proper Time Formalism, Gauge Invariance and the Effects of a Finite World Sheet Cutoff in String Theory

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Abstract

We discuss the issue of going off-shell in the proper time formalism. This is done by keeping a finite world sheet cutoff. We construct one example of an off-shell covariant Klein Gordon type interaction. For a suitable choice of the gauge transformation of the scalar field, gauge invariance is maintained off mass shell. However at second order in the gauge field interaction, one finds that $(U(1))$ gauge invariance is violated due to the finite cutoff. Interestingly, we find, to lowest order, that by adding a massive mode with appropriate gauge transformation laws to the sigma model background, one can restore gauge invariance. The gauge transformation law is found to be consistent, to the order calculated, with what one expects from the interacting equation of motion of the massive field. We also extend some previous discussion on applying the proper time formalism for propagating gauge particles, to the interacting (i.e. Yang Mills) case.

1 Introduction

The sigma-model or renormalization group approach to string theory [1-7] has shown some promise as an alternative to string field theory for doing non-trivial calculations. It is hoped that it will be computationally simpler and that the physical significance of the symmetries be more transparent than in string field theory [8-12]. In the renormalization group approach to string theory there are two outstanding issues. One is that of gauge invariance and the other is that of an off-shell formulation. These two issues are, of course, intertwined because, in general, maintaining gauge invariance off-shell is more difficult than when on shell. The issue of ‘massless’ $U(1)$ gauge invariance in the on-shell sigma model formalism has been discussed some time ago [13,28-30]. Gauge invariance associated with the massive modes is a little more complicated and requires the introduction of an infinite number of ‘proper time’ variables and was discussed in ref[14]. The discussion of gauge invariance in these papers has been restricted to linear gauge transformations determined by the invariance of the free theory. At the interacting level the situation is a lot more complicated, especially off-shell. A discussion of these issues in the BRST formalism is contained in Ref[27].

In ref[15] a study of the gauge invariance of the interacting theory was initiated. We derived, in the proper time formalism [16,21-26] (which is really a variant of the renormalization group approach), the covariant Klein Gordon equation. It was shown that the technique works equally well for point particles as well as strings. For point particles it is exact, while for strings it is derived as a low energy approximation. The usual gauge invariance at the massless level, $\delta A_\mu = \partial_\mu \Lambda$, arises as a freedom to add total derivatives to the two dimensional world sheet action. It was shown that this is no longer a symmetry when interactions are present because of boundary terms. These boundary terms can be cancelled by an appropriate transformation of the Klein Gordon scalar field $\delta\phi = i\Lambda\phi$. This enables us to understand how the ‘interacting’ terms in gauge transformations arise in this formalism. (By ‘interacting terms’ we mean those that are required specifically by the interacting theory). We had also discussed a possible generalization of the proper time formalism when dealing with the propagation of gauge particles. As a first application, we gave (yet another!) derivation of the Maxwell equations.

We would like to extend the results of that work in two directions. One is to study the effect of keeping a finite cutoff on the world sheet. The

motivation for this goes back to Ref[17], where it was argued that in order to go off-shell, in this formalism, one needs a finite cutoff. In the language of the renormalization group this is equivalent to the statement that when the cutoff is finite (w.r.t. the correlation length) one is away from the fixed point (i.e. off-shell) and this is also where the irrelevant operators (i.e. vertex operators for off-shell massive modes) are no longer irrelevant. Thus, in this paper we construct an off shell vertex coupling the photon to two scalars. There is an obvious consistency check. The equation of motion one derives should be an off-shell version of a gauge covariant Klein Gordon equation, and should reduce to the usual one on shell.¹ We should point out that the vertex constructed is not unique - there are many other possibilities. Other constraints have to be imposed to single out one choice.

We then proceed to the next order, which is the cubic term in the covariant Klein Gordon equation - involving two vector fields. Interestingly enough, we find that one can restore gauge invariance (at least to lowest order) by adding a massive “spin-2” particle with an appropriate gauge transformation law. Thus, gauge invariance can be maintained with finite cutoff if we have nontrivial backgrounds for the massive modes. We can also check whether the gauge transformation law for this massive mode is consistent with its (interacting) equation of motion. We find that this is so, at least, to lowest order in momentum. Assuming these results continue to hold to all orders in the cutoff, which will require adding an infinite tower of massive modes, we can say, that, in a sense, keeping a non zero world sheet cutoff is equivalent to keeping all the massive modes. We do not find this statement surprising. Given the renormalization group interpretation, it is to be expected. Nevertheless, we find it very interesting that it is being derived in a completely different way, with no reference to the renormalization group, whatsoever. In ref[14], however, we speculated that this is true for a non zero space time cutoff (rather than just the world sheet cutoff). But we have no calculations to support this speculation, yet.

We also extend the results of ref[15] in another direction. We study the Yang-Mills vector field and use the generalization of the proper-time method to derive its equation of motion (to lowest order in momentum). This is only a vindication of the method described there for gauge particles, since the

¹ Here we use the word ‘off-shell’ to describe fields that do not satisfy the momentum mass shell condition $p^2 = m^2$. They may or may not satisfy the full equations of motion.

result is standard. What would be non trivial is to extend this to a finite cutoff off shell vertex.

This paper is organized as follows: In Section II we present a brief review of the derivation of Maxwell's equation and then derive the Yang Mills case. In Section III we discuss the off shell three point vertex of the covariantised Klein Gordon equation. In Section IV we look at the four point vertex and study the gauge covariance properties. We conclude in Section V. Details of a calculation is given in an Appendix.

2 Equation of Motion for Gauge Fields

The renormalization group equation can be rephrased as a ‘proper time’ equation of the form:

$$\left(\frac{d}{d\ln(z-w)} + 2\right) \langle O(z)O(w) \rangle = 0 \quad (2.1)$$

where $O(z)$ is a vertex operator.

This equation says that $O(z)$ has dimension one. Since $z = e^{\tau+i\sigma}$ where, for open string vertex operators, we have to set $\sigma = 0$, this equation takes the form of a proper time equation familiar for point particles. In Ref[15] we used this to derive the covariant Klein Gordon equation. The expectation value was evaluated with a weight $e^{\frac{1}{2} \int d^2z \partial_z X \partial_{\bar{z}} X + \int dz A_\mu \partial_z X^\mu}$ where A_μ is a background gauge field and $O(z) = \phi[X(z)]$ is the scalar field. We refer the reader to Ref[15] for the details. If instead of $\phi[X(z)]$ we want to consider a gauge field, then $O(z) = A_\mu[X(z)] \partial_z X^\mu$. However this naive substitution is not satisfactory since we need to ensure that the result is invariant under $A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$. The solution proposed in ref[15] is to replace $O(z)$ by $\int dz O(z) = \int dz A_\mu \partial_z X^\mu$. This is invariant, because under a gauge transformation $A \partial X$ changes by a total derivative which vanishes when integrated - assuming that there are no boundaries. However, in this case the operator $\frac{d}{d\ln(z-w)}$ is replaced by the functional derivative $\frac{\delta}{\delta \Sigma(z-w)}$ where $\Sigma(z-w) = \langle X(z)X(w) \rangle$. Thus $\Sigma(z-w)$ is treated as a *field*, rather than as a function.² Thus we have the operation

$$\int dz \int dw \frac{\delta}{\delta \Sigma} \langle O(z)O(w) \rangle = 0 \quad (2.2)$$

$$\Rightarrow \int dz \int dw \frac{\delta}{\delta \Sigma} \langle A_\nu \partial_z X^\nu A_\mu \partial_w X^\mu \rangle = 0 \quad (2.3)$$

with $A_\mu(X) = \int dk A_\mu(k) e^{ikX}$. The coefficient of A_ν gives the equation of motion. To obtain the linear term in the equation of motion one does not need any insertions of $\int dz A_\mu \partial_z X^\mu$ from the sigma model action. So we get:

$$\int dz \int dw \frac{\delta}{\delta \Sigma} [A(p) \cdot k A(k) \cdot p \partial_w \Sigma \partial_z \Sigma + p \cdot k A(p) \cdot A(k) \Sigma \partial_z \partial_w \Sigma] = 0 \quad (2.4)$$

²This is very similar to evaluating derivatives w.r.t. the Liouville mode - something that has been used frequently to derive equations of motion [12,13,14].

$$\Rightarrow [-A(p).kA(k).p + p.kA(p).A(k)]\partial_z\partial_w\Sigma = 0 \quad (2.5)$$

The coefficient of $A_\mu(p)$ is thus

$$-k^\mu A(k).p + p.kA^\mu(k) = 0 \quad (2.6)$$

Thus we get Maxwell's equation. Note, also, that the expression in square brackets in (2.5) is in fact the Maxwell action, (from which (2.6) is obtained by varying w.r.t A_μ) except for a factor of 1/2. This factor of 1/2 is important in the Yang Mills case where we are concerned with the relative normalization vis-a-vis the cubic and quartic terms in the action. Thus, the proper time formalism gives the equations of motion of Yang Mills theory (as we shall see), and not the action. To reconstruct the action, one will have to put in by hand these factors of 1/2, 1/3 or 1/4 (in the cubic and quartic terms respectively).

The crucial point to note is that we are able to integrate by parts on the variable 'w' since we have an integral $\int dw$. This concludes our review.

We now turn to the Yang Mills case where we would like to get the cubic and quartic pieces in a manifestly Lorentz covariant fashion. This calculation can be simplified by the following identity (proved in the Appendix)

$$e^{ik_0X}ik_1\partial X = \int_0^1 d\alpha\partial(e^{i\alpha k_0X}ik_1X) + \int_0^1 d\alpha[e^{i\alpha k_0X}X^\nu\partial X^\mu(i)^2\alpha k_0^{[\nu}k_1^{\mu]}] \quad (2.7)$$

We have written the LHS as a sum of two terms, one of which, involving $F_{\sigma\mu} = \partial_\sigma A_\mu - \partial_\mu A_\sigma$, is manifestly gauge invariant (under $A_\mu \rightarrow A_\mu + \partial_\mu\Lambda$) and the other one is a total derivative that vanishes if there are no boundaries. Furthermore the first term on the RHS can be written as

$$\int_0^1 \frac{d\alpha}{\alpha} \partial(e^{i\alpha(k_0+k_1)X}) \quad (2.8)$$

if we remember to extract the piece that is linear in k_1 .

We can map the upper half plane to a circular disc so that the vertex operators are attached to the circular boundary.

To calculate the cubic term, we have to bring down one factor of $\int A\partial X$ from the exponent. Thus we have to consider

$$\oint dz \oint du < iA_\mu\partial_z X^\mu \int_u^z dw iA_\nu\partial_w X^\nu iA_\rho\partial_u X^\rho > \quad (2.9)$$

Here $A_\mu \equiv A_\mu^a T^a$, where T^a satisfy the Lie Algebra $[T^a, T^b] = if^{abc}T^c$ and $Tr(T^a T^b) = \delta^{ab}$. In correlations we will assume that there is a trace over the matrix indices. Using the notation of ref[14] we will write $A^\mu(X) = \int dk_0 dk_1 k_1^\mu \Phi(k_0, k_1) e^{ik_0 X}$ and for simplicity we will omit the integrals and the field Φ when we write correlations. Thus (2.9) becomes :

$$Tr(T^a T^b T^c) \oint dz \oint du \frac{\delta}{\delta \Sigma} < ik_1^\mu \partial_z X^\mu e^{ik_0 X} i \int_u^z dw p_1^\nu \partial_w X^\nu e^{ip_0 X} i q_1^\rho \partial_u X^\rho e^{iq_0 X} > \quad (2.10)$$

In performing the above calculation we must keep in mind that there is a path ordering, implicitly, for the matrices. Thus the ordering of the matrices inside the trace follows the ordering of the three points u, w and z along the circle. Since the vector bosons are bosons and the resultant interaction is symmetric under permutations, we can restrict ourselves to a particular ordering while evaluating expressions like (2.10), and multiply the result by a combinatoric factor, which is equal to the number of permutations. In the cubic case there is only one insertion, so there are no such factors.

We now substitute the RHS of (2.7) for each of the three factors inside the correlation in (2.10). Amongst the many terms that arise, is the following one:

$$\int_0^1 d\alpha \int_0^1 d\beta \int_0^1 d\gamma \gamma \oint dz \oint du \quad (2.11)$$

$$< ik_1^\mu \partial_z (X^\mu e^{i\alpha k_0 X}) \int_u^z dw i p_1^\nu \partial_w (X^\nu e^{i\beta p_0 X}) i^2 q_0^{[\rho} q_1^{\sigma]} X^\rho \partial_u X^\sigma e^{i\gamma q_0 X} > \quad (2.12)$$

There are also two other terms of this type obtained by interchanging $k \leftrightarrow q$ and $p \leftrightarrow q$. (2.12) now becomes (the α, β and γ integrals are understood):

$$1/2 \oint dz \oint du k_1^\mu p_1^\nu < \partial_z (X^\mu e^{i\alpha k_0 X}) [(X^\nu e^{i\beta p_0 X(z)}) - (X^\nu e^{i\beta p_0 X(u)})] X^\rho \partial_u X^\sigma e^{i\gamma q_0 X} > q_0^{[\rho} q_1^{\sigma]} \quad (2.13)$$

We are interested in the dependence on $\Sigma(z - u)$ in the above expression.³

To obtain the cubic coupling of Yang-Mills theory we only need to keep the lowest order terms in (2.13). In particular the second term inside the

³We ignore the logarithmically divergent terms. The justification for this is that (see [16]) in the final answer $\ln a$ and $\ln(z - u)$ will always occur in the form $\ln(\frac{z-u}{a})$, purely for dimensional reasons. Thus we can concentrate on the $\ln(z - u)$ terms, and set $\ln a$, formally, to zero.

square brackets does not contribute anything, to this order. We can also ignore, to this order, the exponential factors. Thus we get $(\Sigma(z - u) \equiv \Sigma)$:

$$1/2 \oint dz \oint du k_1^\mu p_1^\nu q_0^{[\rho} q_1^{\sigma]} (\delta^{\mu\rho} \delta^{\nu\sigma} \partial_z \Sigma \partial_u \Sigma + \delta^{\mu\sigma} \delta^{\nu\rho} \Sigma \partial_z \partial_u \Sigma) \quad (2.14)$$

Restoring all other factors from (2.9),(2.10) we get

$$1/2 A_\mu^a(k_0) A_\nu^b(p_0) q_0^{[\rho} A_\sigma^c(q_0) 1/2 f^{abc} \oint dz \oint du (\delta^{\mu\rho} \delta^{\nu\sigma} - \delta^{\mu\sigma} \delta^{\nu\rho}) \frac{\delta}{\delta \Sigma} (\Sigma \partial_z \partial_u \Sigma) \quad (2.15)$$

Note that we have integrated by parts on z . In evaluating the matrix trace, we have used antisymmetry in ‘ a ’ and ‘ b ’. The coefficient of $A_\mu(k_0)$ gives the quadratic piece of the Yang-Mills equation of motion. The full symmetry between k, p , and q is manifest when we include the two other terms from eq(2.10) mentioned earlier. As in the case of the free Maxwell case discussed earlier, we see the full cubic term of the Yang-Mills *action* in (2.15) (we also have to add the two other terms necessary for symmetry). Again, as before we have to include a factor of $1/3$ to get the right normalization.

The quartic term can similarly be obtained by calculating:

$$\frac{2!}{2!} \oint dz \oint du \frac{\delta}{\delta \Sigma} < A \partial_z X \int_u^z dw A \partial_w X \int_u^w dv A \partial_v X A \partial_u X > \quad (2.16)$$

The $2!$ in the denominator comes from expanding the exponential and the $2!$ in the numerator comes from having chosen a particular ordering, $w > v$ in (2.16). In the notation of ref[14] this becomes:

$$Tr(T^a T^b T^c T^d) \oint dz \oint du < k_1^\mu \partial_z X^\mu e^{ik_0 X} \int_u^z dw p_1^\nu \partial_w X^\nu e^{ip_0 X} \int_u^w dv q_1^\rho \partial_v X^\rho e^{iq_0 X} l_1^\sigma \partial_u X^\sigma e^{il_0 X} > \quad (2.17)$$

Since we are only interested in the quartic Yang-Mills piece we can set all the momenta to zero. In that case we get on doing the ‘ v ’ integral:

$$\oint dz \oint du < k_1^\mu \partial_z X^\mu \int_u^z dw p_1^\nu \partial_w X^\nu q_1^\rho [X^\rho(w) - X^\rho(u)] l_1^\sigma \partial_u X^\sigma > \quad (2.18)$$

Consider the term involving $X^\rho(w)$. We can rewrite the region of integration (and still preserve the ordering) as $\oint dw \oint du \int_w^u dz$. This gives

$$\oint dw \oint du < l_1^\sigma \partial_u X^\sigma \int_w^u dz k_1^\mu \partial_z X^\mu p_1^\nu \partial_w X^\nu q_1^\rho X^\rho(w) > \quad (2.19)$$

Performing the z integral and evaluating the correlation gives:

$$\oint du \oint dw l_1^\sigma k_1^\mu p_1^\nu q_1^\rho (\delta^{\sigma\nu} \delta^{\mu\rho} \Sigma \partial_u \partial_w \Sigma + \delta^{\sigma\rho} \delta^{\mu\nu} \partial_u \Sigma \partial_w \Sigma) \quad (2.20)$$

Since we can integrate by parts on u or w , we can see that (2.20) is anti-symmetric in σ, μ and ν, ρ indices respectively. Antisymmetry in σ, μ indices implies an antisymmetry in a, d indices. Thus, restoring the group theory factors, we get:

$$1/4 f^{dae} f^{bce} 1/4 l_1^{[\sigma} k_1^{\mu]} p_1^{[\nu} q_1^{\rho]} (\delta^{\sigma\nu} \delta^{\mu\rho} \Sigma \partial_u \partial_w \Sigma + \delta^{\sigma\rho} \delta^{\mu\nu} \partial_u \Sigma \partial_w \Sigma) \quad (2.21)$$

The second term in (2.18) gives a permutation of this. Thus we get (re-expressing in terms of A):

$$\oint du \oint dw [1/2 f^{ade} f^{bce} A_\mu^a(k_0) A_\nu^b(p_0) A_\rho^c(q_0) A_\sigma^d(l_0) (\delta^{\sigma\nu} \delta^{\mu\rho} - \delta^{\sigma\rho} \delta^{\mu\nu})] \frac{\delta}{\delta \Sigma} \Sigma \partial_u \partial_w \Sigma. \quad (2.22)$$

As before, the coefficient of $A(p_0)$ gives the contribution to the equation of motion, and we can recognize the Yang-Mills structure (after appropriate symmetrization). Also as before, we can recognize in the square brackets the quartic term of the Yang-Mills *action*. As mentioned earlier, in going from the equation of motion to the action, we have to divide by 4 in order to get the right normalization. The action that gives the above equation of motion is

$$1/8 [\partial_{[\mu} A_{\nu]}^c + f^{abc} A_\mu^a A_\nu^b]^2 \quad (2.23)$$

The rest of the terms in (2.10) and (2.17) represent higher order string corrections to the above action.

Thus we have demonstrated a method for dealing with interacting gauge particles in the framework of the proper time formalism. The crucial point is to treat the propagator $\langle X(z)X(w) \rangle$ as a field (Σ), and keep the integrals over the coordinates z and w . This allows one to integrate by parts when performing functional differentiation w.r.t Σ . In this paper we have stayed close to the mass-shell. To go off mass-shell in a manifestly gauge covariant way requires a further extension of these methods. In the next section we will address this problem for the simpler case of the Klein-Gordon equation.

3 Going Off-Shell with a Finite Cutoff

If one evaluates

$$\left(\frac{d}{d\ln(z-w)} + 2\right) \langle \phi[X(z)]\phi[X(0)] \rangle > \quad (3.1)$$

with the interaction $\int A_\mu \partial_z X^\mu dz$ in the sigma model action, one gets an equation of motion for ϕ that has arbitrarily high powers of A_μ . There are many ways of understanding this. In terms of the renormalization group, $\int A_\mu \partial_z X^\mu dz$ is a marginal operator (at least for small momenta). In the continuum limit, near a fixed point, the β -function can have terms with arbitrarily large powers of the (marginal) coupling constant and these terms have no suppression factors on dimensional grounds. In terms of Feynman diagrams in string theory, one is looking at tree diagrams with external massless fields, of which, there can be any number. In terms of the equations of motion, the massive fields can be solved for in terms of massless fields, and can be eliminated from the equations.

If one wants to be sensitive to the coefficients of the irrelevant operators and not just the marginal ones, then one should look at distance scales on the order of the underlying lattice cutoff. In this situation the β -functions are finite degree polynomials and involve all the coupling constants. This is the case when one is far from the fixed point. In string theory terms, one is off shell and the massive modes are important and the equations of motion involve all the fields, not just the massless ones. The equations are expected to be polynomial in such a situation. This is the situation described by the cubic string field theory vertex [11].

Thus at the moment we have two extreme situations- the quadratic equations of string field theory and the infinitely non polynomial equations of the sigma model approach. It seems plausible that one should be able to interpolate between these two extremes. The renormalization group interpretation suggests that by varying the value of the underlying cutoff one can modify the degree to which heavy fields are “integrated out”. In ref[16] we showed how this could be done in the proper time formalism: When evaluating eqn. (3.1) we introduce a lattice spacing ‘ a ’ and require that it be the distance of closest approach between two vertex operators. The parameter z/a determines how many vertex operators can be inserted between $\phi[X(z)]$ and $\phi[X(0)]$ and thus the degree of polynomiality of the equation. In particular,

if $z = 2a$ one can insert only one vertex operator and the equations are purely quadratic. If $z/a \rightarrow \infty$ we get a polynomial of arbitrary high order - the sigma model situation. Thus for different values of z/a one gets different sets of equations. One expects that if any of these sets of equations is expressed completely in terms of massless fields, by eliminating the massive ones, we will end up with the equation obtained in the $z/a \rightarrow \infty$ case, i.e. the sigma model case. However we do not yet have a proof of this. We also do not have field theory Feynman rules for calculating higher n-point functions from a given set of lower n-point vertices and propagators. Without such a prescription we cannot really check the consistency of our formulation. This is an important issue that we hope to address in the future. Meanwhile, however, there is one basic consistency check that can be done, and that is to check gauge invariance. We can require that our equation of motion be gauge covariant and that it reduce to the covariant Klein Gordon equation on-shell.

Before we do this, let us note that in order to make contact with critical strings one should be careful about group theory factors. However, in this work we will just discuss a charged string in some background U(1) field. The problem of a charged string moving in an electromagnetic background has been discussed in ref[28]. The point of deviation in our discussion is that we will be keeping a finite cutoff in order to go off-shell. Thus the electromagnetic field can have any momentum dependence.

Thus, following ref[16], we consider the proper time equation

$$z \frac{d}{dz} \left\{ z^2 < e^{ik'X(z)} \int_a^{z-a} du A_\mu[X(u)] \partial_u X^\mu(u) e^{ikX(0)} > \right\} \phi(k) = 0 \quad (3.2)$$

This is the same as eqn (3.1) (with one insertion of A) where $\phi[X(z)]$ has been chosen to be $e^{ik'X(z)}$. The factor z^2 in eqn.(3.2) is needed to produce the second term in eqn.(3.1). Before we proceed to evaluate (3.2) let us evaluate a simpler quantity, namely, the gauge transformation of (3.2) under $A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$. If we replace A_μ by $A_\mu + \partial_\mu \Lambda$ in the covariant Klein-Gordon equation we get $2\partial_\mu \Lambda \partial_\mu \phi + \partial^2 \Lambda \phi$ to lowest order. We would first like to make sure that we reproduce this. We get:

$$z \frac{d}{dz} \left\{ z^2 < e^{ik'X(z)} [\Lambda(q) e^{iqX(z-a)} - \Lambda(q) e^{iqX(a)}] e^{ikX(0)} > \right\} \phi(k) = 0 \quad (3.3)$$

where we have gone over to the momentum representation for A and Λ .

$$z \frac{d}{dz} [a^{k'.q} z^{k'.k+2} (z-a)^{q.k} - (z-a)^{k'.q} z^{k'.k+2} a^{q.k}] \phi(k) = 0 \quad (3.4)$$

This gives

$$[(k'.k+2)a^{k'.q} z^{k'.k+2} (z-a)^{q.k} + q.k a^{k'.q} z^{k'.k+3} (z-a)^{q.k-1} \quad (3.5)$$

$$-(k'.k+2)(z-a)^{k'.q} z^{k'.k+2} a^{q.k} - k'.q (z-a)^{k'.q-1} z^{k'.k+3} a^{q.k}] \phi(k) = 0$$

On-shell, setting $k.k' + 2 = q.k = q.k' = 0$ in the exponents, we get the leading order ⁴ contribution:

$$(q.k - q.k') \frac{z}{z-a} \Lambda(q) \phi(k) = (q^2 + 2q.k) \Lambda(q) \phi(k) \frac{z}{z-a} = 0 \quad (3.6)$$

where we have used $k + k' + q = 0$. What it should give on-shell is, of course, $(q^2 + 2q.k) \Lambda(q) \phi(k) = 0$, which is just the change, under $\delta\phi = i\Lambda\phi$, of the Klein Gordon equation. We can get rid of the factor $\frac{z}{z-a}$ by replacing the factor z^2 in eq. (3.2) by $z(z-a)$. We will do this from now on. In the limit $a \rightarrow 0$, this does not make any difference. But for finite a , the proper time equation is modified. Thus, we evaluate (3.2) with $A_\mu[X(u)] = \int dq A_\mu(q) e^{iq.X(u)}$ and z^2 replaced by $z(z-a)$ and find the result

$$\begin{aligned} & \{ (A.k + \frac{A.k'}{q.k'}) (x-1)^{q.k} x^{k'.k+2} - \\ & (A.k' + \frac{A.k'}{q.k'}) (x-1)^{q.k'} x^{k'.k+2} + \\ & (k'.k+1) \frac{A.k'}{q.k'} (x-1)^{q.k+1} x^{k'.k+1} - \\ & (k.k'+1) \frac{A.k'}{q.k'} (x-1)^{q.k'+1} x^{k'.k+1} + \end{aligned} \quad (3.7)$$

⁴ Upto overall powers of a , which can be gotten rid of by renormalizing the vertex operators and using momentum conservation : each vertex operator comes with a factor a raised to the scaling dimension of the operator, which in turn is equal to $k^2/2$ + number of derivatives in the vertex operator.

$$(x-1)(q^2 A^\mu - q \cdot A q^\mu) k_\mu \int_1^{x-1} du (x-u)^{k' \cdot q - 1} u^{q \cdot k - 1} x^{k' \cdot k + 2} + \frac{(q^2 A^\mu - q \cdot A q^\mu)}{q \cdot k'} k_\mu [(k \cdot k' + 1)(x-1) + x] \int_1^{x-1} du (x-u)^{k' \cdot q} u^{q \cdot k - 1} x^{k' \cdot k + 1} \} \phi(k) = 0$$

The last two terms vanish when the photon is on-shell (i.e. $\partial_\mu F_{\mu\nu} = 0$). We also note that the appearance of a pole at $k' \cdot q = 0$ is deceptive, since the pole terms cancel out. If we choose $x = 2$, the integrals vanish, and we get

$$(A \cdot k - A \cdot k') 2^{k' \cdot k + 2} \phi(k) = (2A \cdot k + A \cdot q) 2^{-k^2 - k \cdot q + 2} \phi(k) = 0 \quad (3.8)$$

Furthermore, on-shell, if we set $k' \cdot k + 2 = q \cdot k = q \cdot k' = 0$, (3.7) reduces to

$$(A \cdot k - A \cdot k') \phi(k) = (2A \cdot k + A \cdot q) \phi(k) = 0 \quad (3.9)$$

In this case we can also let $a \rightarrow 0$ or $x \rightarrow \infty$. Thus, the main result of this section is the contribution to the Klein Gordon equation given by (3.7). The integrals can be expressed in terms of hypergeometric functions, but we will not do so here.

On the face of it expression (3.7) looks like yet another 3-pt. vertex that should be obtainable from some string field theory [11,12]. However, when we study (3.2) and (3.7) we see a difference. In (3.2) there is an integral over u , the location of A_μ . The vertices considered in the literature thus far always have the three vertex operators in specific locations. In the special case of $z = 2a$ these integrals (in eqn. (3.7)) vanish, and we have well defined locations for the vertex operators.

We also have a rule that ‘ a ’ is the minimum spacing between two vertex operators. Thus, for instance, when we consider the gauge transformed kinetic term in the equation of motion, we get

$$z \frac{d}{dz} [z(z-a) < \phi[X(z)] (\Lambda[X(z-a)] - \Lambda[X(a)]) \phi[X(0)] >] = 0 \quad (3.10)$$

The important point is that

$$\delta \phi[X(0)] = i \Lambda[X(a)] \phi[X(0)] \quad (3.11)$$

and not

$$\delta \phi[X(0)] = i \Lambda[X(0)] \phi[X(0)] \quad (3.12)$$

In fact, we have seen that when we substitute $A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$ we get eqn. (3.3) (with z^2 replaced by $z(z-a)$), which is identical with (3.10). Thus it is the transformation law (3.11) that is consistent with the gauge invariance of (3.7). However, until we have a full formulation that spells out the precise relation between the (N+1)-point function and the N-point function we cannot claim that this rule of keeping a minimum spacing ‘ a ’ between vertex operators is consistent.

If we choose $z = 2a$, then the equation of motion is quadratic in the fields. In this case the variation $\delta\phi = -i\Lambda\phi$ of the $A\phi$ piece has to be cancelled by the variation of a massive mode rather than by the variation of the $A-A-\phi$ piece as in point particle field theory. If $z/a > 3$ one can have two or more powers of A . In this case one expects a cubic $A-A-\phi$ term in the equation of motion. The gauge transformation property of this term is the topic of the next section.

There are also other modifications that are possible. One can imagine doing the same calculation on a disc where cyclic symmetry is manifest. This would be similar to what is done in ref[18,19,20]. In this case the equation of motion would be different.

Our main aim in this paper is to explore the issues that arise in keeping a finite cutoff - particularly issues of gauge invariance. We should keep in mind that geometries other than that used in eq (3.7) are possible. The results of ref[11,12], in fact, suggest that manifest cyclic symmetry is very important for the full gauge invariance of the theory.

4 Gauge Invariance and Finite Cutoff

We have been using gauge invariance as a consistency check on the off-shell terms in the Klein Gordon equation. At the next order in A_μ something interesting happens: effects of a finite cutoff show up even at the classical level (i.e. before doing the functional integral over $X(z)$). To see this, consider the second order term in the proper time equation:

$$< e^{ik'X(z)} \int_{2a}^{z-a} du A_\mu[X(u)] \partial_u X^\mu(u) \int_a^{u-a} dw A_\nu[X(w)] \partial_w X^\nu(w) e^{ikX(0)} > \quad (4.1)$$

When we perform a gauge variation we get:

$$\begin{aligned} & < e^{ik'X(z)} \Lambda[X(z-a)] \int_a^{z-2a} dw A_\nu[X(w)] \partial_w X^\nu(w) e^{ikX(0)} > \quad (4.2) \\ & - < e^{ik'X(z)} \int_{2a}^{z-a} du A_\nu[X(u)] \partial_u X^\nu(u) \Lambda[X(a)] e^{ikX(0)} > \\ & - < e^{ik'X(z)} \int_{2a}^{z-a} du \Lambda[X(u)] A_\nu[X(u-a)] \partial_u X^\nu(u-a) e^{ikX(0)} > \\ & + < e^{ik'X(z)} \int_{2a}^{z-a} du A_\nu[X(u)] \partial_u X^\nu(u) \Lambda[X(u-a)] e^{ikX(0)} > . \end{aligned}$$

Note that we have consistently imposed the rule that ‘ a ’ is the distance of closest approach between two vertex operators by appropriate choice of the limits of integration in (4.2). Notice that the first two terms in expression (4.2) can be cancelled in the usual way by a variation $\delta\phi = -i\Lambda\phi$ on the first order term considered in the previous section. The other two terms, however, remain to be cancelled. The third and fourth terms in (4.2) would cancel if ‘ a ’ were equal to zero. Their sum is thus proportional to ‘ a ’. and can be written as the sum of three terms:

$$\begin{aligned} & \int_a^{z-a} du A_\nu[X(u)] \partial_u X^\nu(u) [\Lambda[X(u-a)] - \Lambda[X(u+a)]] \quad (4.3) \\ & - \int_a^{2a} du A_\nu[X(u)] \partial_u X^\nu(u) [\Lambda[X(u-a)]] \\ & + \int_{z-2a}^{z-a} du [\Lambda[X(u+a)] A_\nu[X(u)] \partial_u X^\nu(u) \end{aligned}$$

The first term is what we get by modifying the limits of integration of the third and fourth terms in (4.2) so that they both go from a to $z - a$. The remaining terms (of (4.3)) compensate the $O(a)$ errors that arise from such a modification (of limits). Each of the terms in (4.3) can, in turn, be expanded in powers of ‘ a ’. The lowest order terms are :

$$\int_a^{z-a} du A_\nu[X(u)] \partial_u X^\nu(u) (-2a) \partial_u \Lambda[X(u)] \quad (4.4)$$

$$\begin{aligned} & -a A_\mu \partial_u X^\mu(a) \Lambda(a) + a \Lambda(z-a) A_\mu \partial_u X^\mu(z-a) \\ & = -2a \int_a^{z-a} du A_\mu \partial_\nu \Lambda \partial_u X^\mu \partial_u X^\nu - a [A_\mu \partial_u X^\mu(a) \Lambda(a) - \Lambda(z-a) A_\mu \partial_u X^\mu(z-a)] \end{aligned} \quad (4.5)$$

The first term in (4.5) is a ‘bulk’ term and the second one is a boundary term. The first term, in fact, looks like the vertex operator for a massive mode. Thus, consider a massive field $S_{\mu\nu}$ (see eq. (4.8)) with the transformation law:

$$\delta S_{\mu\nu} = 2a(A_\mu \partial_\nu \Lambda + A_\nu \partial_\mu \Lambda) \quad (4.6)$$

We could add the background field $a \int S_{\mu\nu} \partial_u X^\mu \partial_u X^\nu du$ to the sigma-model action, (along with $\int A_\mu \partial_u X^\mu du$) and the gauge variation of the first order contribution to the proper time equation:

$$< e^{ik'X(z)} a \int_a^{z-a} du S_{\mu\nu} \partial_u X^\mu \partial_u X^\nu e^{ikX(0)} > \quad (4.7)$$

would cancel the first term in (4.5). The boundary term in (4.5) can also be cancelled as follows: Consider, again, the second mass level in string theory. In addition to $S_{\mu\nu}$, there is an auxiliary field S_μ , and the complete vertex operator is [14]:

$$I = a \int dz \frac{1}{2} S_{\mu\nu} \partial X^\mu \partial X^\nu + S_\mu \partial^2 X^\mu \quad (4.8)$$

and the usual ‘free’ gauge transformation of string theory is [14]:

$$\delta S_{\mu\nu} = \partial_\mu \Lambda_\nu + \partial_\nu \Lambda_\mu \quad \delta S_\mu = \Lambda_\mu \quad (4.9)$$

Under these transformations

$$\delta I = a \int_c^d dz [\partial \Lambda_\nu \partial X^\nu + \Lambda_\nu \partial^2 X^\nu] = a \int_c^d dz \partial_z [\Lambda_\nu \partial_z X^\nu] \quad (4.10)$$

$$= a(\Lambda_\nu \partial_z X^\nu(d) - \Lambda_\nu \partial_z X^\nu(c))$$

Clearly if we choose

$$\Lambda_\nu = -A_\nu \Lambda \quad (4.11)$$

we can cancel the second term in (4.5). Thus, we conclude that, at least classically, and to lowest order, gauge invariance can be restored by the addition of massive modes with an appropriate gauge transformation law. In fact, one can even say, based on the above, that imposing the U(1) gauge invariance associated with the *massless* vector, on a theory with *finite* world sheet cutoff requires the presence of *massive* fields. We find this very interesting.

Now, the validity of the Taylor expansion, when quantum mechanics is turned on, needs to be checked. We go back to (4.3) and consider the first term. Perform an operator product expansion on each piece separately:

$$A_\mu(p) : e^{ipX(u)} \partial_u X^\mu(u) : \Lambda(q) : e^{iqX(u-a)} := \quad (4.12)$$

$$A_\mu(p) \Lambda(q) : \left(\frac{iq^\mu}{a} + \partial_u X^\mu \right) | a |^{p,q} e^{i(p+q)X(u)} (1 - iaq \cdot \partial_u X + \dots) :$$

$$A_\mu(p) : e^{ipX(u)} \partial_u X^\mu(u) : \Lambda(q) : e^{iqX(u+a)} := \quad (4.13)$$

$$A_\mu(p) \Lambda(q) : \left(\frac{iq^\mu}{-a} + \partial_u X^\mu \right) | a |^{p,q} e^{i(p+q)X(u)} (1 + iaq \cdot \partial_u X + \dots) :$$

Subtracting (4.13) from (4.12), we get for the first term in (4.3):

$$A_\mu(p) \Lambda(q) : \left(\frac{2iq^\mu}{a} - 2iaq \cdot \partial_u X \partial_u X^\mu \right) | a |^{p,q} e^{i(p+q)X(u)} : \quad (4.14)$$

This corresponds to the operator

$$[-2a : \partial_\nu \Lambda A_\mu \partial_u X^\mu \partial_u X^\nu : + \frac{2}{a} : A^\mu \partial_\mu \Lambda :] | a |^{p,q} \quad (4.15)$$

(If we assume that a normal ordering factor $a^{\frac{p^2}{2}}$ accompanies each exponential e^{ipX} , then, the factor a^{pq} in (4.15) is part of the normal ordering associated with $e^{i(p+q)X}$). The first term in (4.15) is what we got by the Taylor expansion in eqn. (4.5). The second term is a normal ordering effect - it is a contraction of the two $\partial_u X$'s. It corresponds to a tachyon like operator

and thus represents a gauge transformation on the background tachyon field :

$$\delta\phi_T = -2A_\mu\partial_\mu\Lambda \quad (4.16)$$

We can do a similar analysis for the boundary terms and one finds that in addition to the terms in (4.5) one needs a term of the form:

$$\int du\partial_u(A_\mu\partial_\mu\Lambda) = \int du\partial_\nu(A_\mu\partial_\mu\Lambda)\partial_uX^\nu \quad (4.17)$$

This is clearly a U(1) gauge transformation of A_μ itself with parameter $A_\mu\partial_\mu\Lambda$. We have thus shown that the boundary terms can be compensated by usual linear gauge transformations of $S_{\mu\nu}, S_\mu$ and A_μ merely by redefining the gauge parameter.⁵ Thus we conclude that (to this order) the theory is gauge invariant with a finite cutoff provided we modify the gauge transformations of the massive spin 2, the vector, and of the tachyon in well defined ways.

There is one consistency check that can be quickly performed. One can check whether the non-linear pieces introduced in the gauge transformation law of $S_{\mu\nu}, \phi_T$ are consistent with their interacting equations of motion. Obtaining the fully covariant interacting equations of motion is a complicated story. This has been done in some detail in [30], in the case where the electromagnetic field is slowly varying in space-time, which is all we need to lowest order. One can also write down the leading pieces (lowest order in momentum) just by considering the OPE of $A_\mu\partial_zX^\mu$ with itself:

$$\begin{aligned} A_\mu(p) : \partial_zX^\mu(z)e^{ipX(z)} : A_\nu(q) : \partial_zX^\nu(z+a)e^{iqX(z+a)} : = \\ A_\mu(p)A_\nu(q) : \partial_zX^\mu(z)\partial_zX^\nu(z+a)e^{ipX(z)+iqX(z+a)} : | a |^{p,q} + \\ A.A\frac{1}{a^2} : e^{ipX(z)+iqX(z+a)} : | a |^{p,q} + \text{higher order in } p, q \end{aligned} \quad (4.18)$$

Thus to lowest order in momentum:

$$= [A_\mu(x)A_\nu(x)\partial_zX^\mu\partial_zX^\nu + O(a)] + \frac{A(x).A(x)}{a^2} + O(a) \quad (4.19)$$

Thus the equation of motion of $S_{\mu\nu}$, which starts out as $(\frac{p^2}{2} - 1)S_{\mu\nu}$ gets modifies to

$$(\frac{p^2}{2} - 1)S_{\mu\nu} + 2A_\mu A_\nu \approx -S_{\mu\nu} + 2A_\mu A_\nu \approx 0 \quad (4.20)$$

⁵It should be pointed out, that an alternative way is to modify the transformation of ϕ . If we add a piece $\delta\phi = iA_\mu\partial_\mu\Lambda\phi$ it would do just as well.

and the equation for ϕ becomes :

$$(\frac{p^2}{2} + 1)\phi + A_\mu A^\mu \approx \phi + A.A \approx 0 \quad (4.21)$$

Both these, (4.20) and (4.21) are consistent with the modifications (4.6) and (4.16) respectively.

Thus we conclude that keeping a finite cutoff while retaining gauge invariance forces one to introduce background massive modes, and to modify their transformation laws. These modifications are consistent with what one expects from their interacting equations. We expect that at higher order in ‘ a ’ one will need other massive modes as well. Now, all this is not surprising. A string is a non local object, (but with local interactions) which is why we have an infinite number of point particles. A finite cutoff makes the theory non-local, albeit in a crude way. Requiring gauge invariance is a way to make this more refined and ‘string’-like. This requires massive modes. One can also argue, as in the introduction, from the viewpoint of the renormalization group, that keeping a finite cutoff entails retaining all the irrelevant operators that correspond to massive modes. Thus there are different ways to understand or rationalize these results. Nevertheless we think that deducing the existence of massive modes and their transformation properties from the requirement of ordinary (‘massless’) gauge invariance, is very interesting.

5 Conclusions

In this paper we have investigated two interrelated topics i) gauge invariance at the interacting level , ii) keeping a finite cutoff and going off-shell. We have a technique for dealing with gauge particles in the proper-time framework. This was an extension to vector-vector interactions of the results of ref[15] for free gauge particles. We do not yet know how to extend this to an off shell calculation. In section 3 we discussed the simpler version of the above problem: going off-shell with a finite cutoff in the case of the covariant Klein Gordon equation. We presented one possible form of the interacting term that satisfies some basic properties of gauge invariance and of having the right on-shell limit. There are other solutions possible. In particular, if one does the same calculation, but on the boundary of a disc, one will have manifest cyclic symmetry. In order to proceed further, one needs a prescription for going from 3-point functions to 4-point functions or higher n-point functions. This, we think, is the most pressing issue in this approach. In sec. 4, in studying the 4-point function directly, we discovered that a non-zero cutoff, along with the requirement of gauge invariance, predicts not only the existence of massive modes, but also the right transformation law. We find this promising. It would be interesting to extend these results to all the massive modes and higher invariances. Finally, on a more speculative note, the idea of a finite world sheet cutoff has to get translated to a finite space-time cutoff.

A Appendix

We will prove eqn.(2.7) by proving the following relation:

$$\frac{(ik_0.X)^n}{n!}ik_1\partial X = \frac{1}{(n+1)!}\partial((ik_0X)^nik_1.X) + \frac{n}{(n+1)!}((ik_0X)^{n-1}X^\sigma\partial X^\mu(i)^2k_0^{[\sigma}k_1^{\mu]}) \quad (\text{A.1})$$

L.H.S. of (A.1) obviously sums to $e^{ik_0X}ik_1\partial X$. The second term is

$$\begin{aligned} \frac{1}{(n+1)}\partial\left(\frac{(ik_0X)^n}{n!}ik_1.X\right) &= \int_0^1 d\alpha \alpha^n \partial\left(\frac{(ik_0X)^n}{n!}ik_1.X\right) = \\ &= \int_0^1 d\alpha \partial\left(\frac{(i\alpha k_0X)^n}{n!}ik_1.X\right) \end{aligned} \quad (\text{A.2})$$

which sums to $\int_0^1 d\alpha \partial(e^{i\alpha k_0X}ik_1X)$. Similarly the last term becomes

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{n}{(n+1)!}((ik_0X)^{n-1}X^\sigma\partial X^\mu(i)^2k_0^{[\sigma}k_1^{\mu]}) &= \\ = \int_0^1 d\alpha \sum_{n=1}^{\infty} \frac{1}{(n-1)!}((i\alpha k_0X)^{n-1}X^\sigma\partial X^\mu(i)^2\alpha k_0^{[\sigma}k_1^{\mu]}) &= \\ = \int_0^1 d\alpha e^{i\alpha k_0X}X^\sigma\partial X^\mu(i)^2\alpha k_0^{[\sigma}k_1^{\mu]} \end{aligned} \quad (\text{A.3})$$

Thus we have to prove (A.1), which we shall do by recursion: Consider the relation

$$\begin{aligned} X^{\mu_1}X^{\mu_2}\dots X^{\mu_{n-1}}\partial X^{\mu_n} &= \frac{1}{n}\partial(X^{\mu_1}X^{\mu_2}\dots X^{\mu_n}) + \\ \frac{1}{n} \underbrace{[X^{\mu_1}X^{\mu_2}\dots X^{[\mu_{n-1}}\partial X^{\mu_n]} + X^{\mu_{n-1}}X^{\mu_1}X^{\mu_2}\dots X^{[\mu_{n-2}}\partial X^{\mu_n]} + \dots]}_{n-1 \text{ cyclic permutations}} \end{aligned} \quad (\text{A.4})$$

Multiply by $(i)^n k_0^{\mu_1} k_0^{\mu_2} \dots k_0^{\mu_{n-1}} k_1^{\mu_n}$ to get

$$\begin{aligned} (ik_0X)^{n-1}(ik_1\partial X) &= \frac{1}{n}\partial((ik_0X)^{n-1}ik_1X) + \\ \frac{1}{n} \underbrace{[(ik_0X)^{n-2}X^{[\mu_{n-1}}\partial X^{\mu_n]}i^2k_0^{\mu_{n-1}}k_1^{\mu_n} + [(ik_0X)^{n-2}X^{[\mu_{n-2}}\partial X^{\mu_n]}i^2k_0^{\mu_{n-2}}k_1^{\mu_n} + \dots]}_{n-1 \text{ terms}} \end{aligned} \quad (\text{A.5})$$

$$\begin{aligned} \Rightarrow (ik_0 X)^{n-1} (ik_1 \partial X) &= \frac{1}{n} \partial((ik_0 X)^{n-1} ik_1 X) + \\ &\quad \frac{n-1}{n} (ik_0 X)^{n-2} X^\sigma \partial X^\mu k_0^{[\sigma} k_1^{\mu]} i^2 \end{aligned} \quad (\text{A.6})$$

Dividing by $(n-1)!$ and replacing n by $n+1$ gives (A.1). Thus it remains to prove (A.4).

We have

$$\begin{aligned} X^{\mu_1} X^{\mu_2} \dots X^{\mu_n} \partial X^{\mu_{n+1}} &= \\ X^{\mu_1} \frac{1}{n} \underbrace{[\partial(X^{\mu_2} \dots X^{\mu_{n+1}}) + X^{\mu_2} X^{\mu_3} \dots X^{[\mu_n} \partial X^{\mu_{n+1}]} + X^{\mu_n} X^{\mu_2} \dots X^{[\mu_{n-1}} \partial X^{\mu_{n+1}]} + \dots]}_{n-1 \text{ cyclic permutations}} \end{aligned} \quad (\text{A.7})$$

where we have used (A.4).

$$\begin{aligned} &= \frac{1}{n} 1/2 (X^{\mu_1} \partial(X^{\mu_2} \dots X^{\mu_{n+1}}) + \partial X^{\mu_1} (X^{\mu_2} \dots X^{\mu_{n+1}})) + \\ &\quad 1/2 (X^{\mu_1} \partial(X^{\mu_2} \dots X^{\mu_{n+1}}) - \partial X^{\mu_1} (X^{\mu_2} \dots X^{\mu_{n+1}})) + \\ &\quad \frac{1}{n} [X^{\mu_1} \underbrace{X^{\mu_2} X^{\mu_3} \dots X^{[\mu_n} \partial X^{\mu_{n+1}]}]}_{n-1 \text{ perm. of } \mu_2 \dots \mu_n}] \\ &= \frac{1}{n} [1/2 \partial(X^{\mu_1} \dots X^{\mu_{n+1}}) + 1/2 (X^{\mu_1} X^{\mu_2} \partial X^{\mu_3} \dots X^{\mu_{n+1}} + \dots + X^{\mu_1} \dots X^{\mu_n} \partial X^{\mu_{n+1}})] + \\ &\quad \frac{1}{n} [X^{\mu_1} \underbrace{X^{\mu_2} X^{\mu_3} \dots X^{[\mu_n} \partial X^{\mu_{n+1}]}]}_{n-1 \text{ perm. of } \mu_2 \dots \mu_n}] \end{aligned} \quad (\text{A.8})$$

In going from (A.8) to (A.9) we have dropped a term manifestly antisymmetric in $\mu_1 \mu_2$ since the LHS is manifestly symmetric. From (A.7) and (A.9) we have:

$$\begin{aligned} (1 - \frac{1}{2n}) X^{\mu_1} X^{\mu_2} \dots X^{\mu_n} \partial X^{\mu_{n+1}} &= \frac{1}{n} [1/2 \partial(X^{\mu_1} \dots X^{\mu_{n+1}}) + \\ &\quad + 1/2 (X^{\mu_1} X^{\mu_2} \partial X^{\mu_3} \dots X^{\mu_{n+1}} + \dots + X^{\mu_1} \dots \partial X^{\mu_n} X^{\mu_{n+1}})] + \\ &\quad + \frac{1}{n} [X^{\mu_1} \underbrace{X^{\mu_2} X^{\mu_3} \dots X^{[\mu_n} \partial X^{\mu_{n+1}]}]}_{n-1 \text{ perm. of } \mu_2 \dots \mu_n}] \end{aligned} \quad (\text{A.10})$$

We can now add all cyclic permutations of μ_1, \dots, μ_n . The LHS is manifestly symmetric and we get a factor of n . The resultant equation is: (We have divided (A.10) by $\frac{2n-1}{2n}$)

$$\begin{aligned}
nX^{\mu_1}X^{\mu_2}\dots X^{\mu_n}\partial X^{\mu_{n+1}} &= \frac{n}{(2n-1)}\partial(X^{\mu_1}\dots X^{\mu_{n+1}}) + \\
&\frac{n-2}{2n-1}\partial(X^{\mu_1}\dots X^{\mu_n})X^{\mu_{n+1}} \\
&+ \frac{2(n-1)}{2n-1}\underbrace{[X^{\mu_1}\dots X^{\mu_n}\partial X^{\mu_{n+1}}]}_{n \text{ perm}}
\end{aligned} \tag{A.11}$$

Combining terms we get

$$\begin{aligned}
(2n^2 - 2)X^{\mu_1}X^{\mu_2}\dots X^{\mu_n}\partial X^{\mu_{n+1}} &= 2(n-1)\partial(X^{\mu_1}\dots X^{\mu_{n+1}}) + \\
&2(n-1)\underbrace{[X^{\mu_1}\dots X^{\mu_n}\partial X^{\mu_{n+1}}]}_{n \text{ perm}}
\end{aligned} \tag{A.12}$$

Dividing throughout by $2n^2 - 2$ gives us a relation that is the same as (A4) with n replaced by $n + 1$. This proves the recursion. Since $X^\rho\partial X^\mu = 1/2\partial(X^\rho X^\mu) + 1/2X^{[\rho}\partial X^{\mu]}$, the relation is true for $n = 2$ and this completes the proof.

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